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Spin–fermion mappings for even Hamiltonian operators

Alberto Anfossi and Arianna Montorsi

Dipartimento di Fisica del Politecnico di Torino, 10129 Torino, Italy

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Abstract

We revisit the Jordan–Wigner transformation, showing that—rather than a non-local isomorphism between different fermionic and spin Hamiltonian operators—it can be viewed in terms of *local* identities relating different realizations of projection operators. The construction works for arbitrary dimension of the ambient lattice, as well as of the on-site vector space, generalizing Jordan–Wigner's result. It provides a direct mapping of local quantum spin problems into local fermionic problems (and vice versa), under the (rather physical) requirement that the latter are described by Hamiltonians which are even products of fermionic operators. As an application, we specialize to mappings between constrained-fermion models and spin-1 models on chains, obtaining in particular some new integrable spin Hamiltonian, and the corresponding ground-state energies.

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1. Introduction

Recently [1, 2] increasing interest has been given to the problem of mapping quantum lattice models of interacting spins into fermionic lattice models, and vice versa, in the spirit of unravelling hidden structures (symmetries) of the problem by changing its representation, and possibly identifying new integrable cases. This is relevant for instance in the study of quantum phase transitions, i.e. zero-temperature changes of macroscopic order induced by some interaction parameter.

The idea of spin–fermion mapping relies in fact on the old result by Jordan and Wigner [3], who first transformed quantum spin $S = 1/2$ operators, which commute at different lattice sites, into operators obeying a Clifford algebra (fermions); the transformation was used for mapping the one-dimensional XX model into a spinless fermion model, the latter being exactly solvable. The Jordan–Wigner transformation (JWT) was recently generalized in [1] to the cases of arbitrary spin S, which are naturally mapped into multi-flavoured fermions (for instance, electrons with spin).

It is interesting to note that the JWT is always a *non-local* transformation: in order to change the algebra of the single-particle operators at a given site (which is known as transmutation of statistics [2]), the transformation in fact involves products of non-trivial operators at each lattice site. Nonetheless the JWT—being usually applied to Hamiltonians which are (sums of) even products of single-particle operators—turns out to transform *local* Hamiltonians into each other; here, given a D-dimensional lattice Λ with L sites, we define *locally* any Hamiltonian $\mathcal{H}_{\mathbf{j}}$ acting on n neighbouring sites of $\mathbf{j}$ in Λ, such that $\lim_{L\to\infty}\frac{n}{L}=0$. JWT—when applied to physically meaningful global Hamiltonian operators $\mathcal{H}\doteq\sum_{\mathbf{j}}\mathcal{H}_{\mathbf{j}}$—always maps a local spin–spin term (say $\mathcal{H}_{\mathbf{j}}^{(S)}$) into a local electron–electron term at the same lattice site $\mathcal{H}_{\mathbf{j}}^{(F)}$. Such an observation suggests that the mapping induced by the non-local JWT could fruitfully be generated by just a *local* transformation.

The spirit of the present paper is to unravel such local transformation, and to give a systematic prescription which allows us to obtain directly at a local level both the results of (generalized) JWT and new mappings between interacting spin and electron Hamiltonian operators. This is useful for instance when looking for integrable one-dimensional models: it is well known that $\mathcal{H}$ does correspond to an integrable model whenever the local matrix representing $\mathcal{H}_{\mathbf{j}}$ can be expressed as a derivative of a R-matrix satisfying appropriate equations (Yang–Baxter equations, see [4] and references therein). Hence integrability amounts to a local property: understanding the local nature of the JWT should allow us to generate from a given local R-matrix both fermionic and spin integrable models.

In order to achieve this goal, we first focus our attention on the matrix representation of the 'isomorphic' operators related by the JWT: the crucial and trivial observation is that they have in fact identical matrix representation, meaning that they can be viewed as different realizations—in terms of spin and fermionic operators respectively—of a unique formal operator. The latter step will be achieved in the following through the combined use of on-site projection operators, and of the theory of matrix representation for graded operators (see for instance [5] and references therein). After introducing the reader to the method we then specialize to one-dimensional case, obtaining a single simple local equation relating spin S models to multi-flavoured fermionic models, for arbitrary S. This is the main result of our paper; the latter is shown to reproduce the results of JWT for spin $1/2$ systems. We then explore in some detail the correspondence between constrained fermions models and spin $S=1$ models; in particular, by this analysis we obtain some new integrable spin-1 model, for which we explicitly derive the ground-state energy.

2. Matrix representation for even Hamiltonian operators

In order to obtain a matrix representation for a given Hamiltonian operator $\mathcal{H}$, we have to specify the global vector space $V^{(\mathrm{glob})}$ on which $\mathcal{H}$ acts. We begin by stating that this space is a tensor product of L copies of the local vector space V at each lattice site,

$$V^{(\mathrm{glob})}=\underbrace{V\otimes\cdots\otimes V}_{L\ \text{times}}. \tag{1}$$

Here the order of the sites associated with the different copies of V has to be fixed, meaning that the sites $\mathbf{j}$ on the D-dimensional lattice have to be put in one-to-one correspondence with a scalar number j ranging from 1 to L. While such correspondence is quite natural for $D=1$ (where $\mathbf{j}\equiv j$), there are many possible different choices in the case of dimension greater than one (see for instance [2]). We shall not pursue this argument further here, and in what follows we simply assume that such correspondence has been set.

Moreover, assuming that V has dimension d, we denote by $|\alpha_j\rangle$ the d state vectors which span V at site j. These are defined through d formal raising operators $\{h_j^{(\alpha_j)}\}$ which act on the local vacuum $|0\rangle$, $|\alpha_j\rangle \doteq h_j^{(\alpha_j)}|0\rangle$. At this level, looking for different realizations of $\mathcal{H}$ reduces to looking for different realizations of the $h_j^{(\alpha_j)}$. In the following we shall denote the different realizations by an index X which can assume both values $X = S$ for spin realizations, and $X = F$ for fermionic ones; whereas, whenever we omit such an index, we refer to the abstract operators, i.e. X can assume both values. For instance, for spin realization, V_S is spanned by the vacuum and the $d-1$ non-vanishing powers of the raising operator S_+ of a spin $\mathbf{S}$ operator with eigenvalue $S = (d-1)/2$; here the following usual $su(2)$ commutation relations hold,

$$\left[S_+^{(j)}, S_-^{(j')}\right] = \delta_{j,j'} 2S_z^{(j)}, \qquad \left[S_z^{(j)}, S_\pm^{(j')}\right] = \pm\delta_{j,j'} S_\pm^{(j)}, \tag{2}$$

whereas, in the case of fermionic realization with $d = 2^f$ (with $f \in \mathbf{N}$ number of flavours of the fermions), V_F will be spanned by even and odd products of f fermionic creation operators $c_{j,s}^\dagger$, which satisfy the Clifford algebra with anticommutation relations given by

$$\left\{c_{j,s}^\dagger, c_{j',s'}^\dagger\right\} = 0, \qquad \left\{c_{j,s}^\dagger, c_{j',s'}\right\} = \delta_{j,j'}\delta_{s,s'}; \qquad s, s' = 1, \ldots, f. \tag{3}$$

Due to the different (anti)-commutation relations for the operators which realize V, the latter may (V_F) or may not (V_S) have an intrinsic graduation; in particular, $V_F = V^{(0)} \bigoplus V^{(1)}$, where the odd (even) subspace $V^{(1)}$ ($V^{(0)}$) is spanned by those vectors that are built with an odd (even) number of creation operators. Similarly, vectors and operators are said to have a parity $p = 1$ ($p = 0$).

With the above specifications we can write the basis vector of the global vector space $V^{(\mathrm{glob})}$ as

$$|\alpha_1, \ldots, \alpha_L\rangle \stackrel{\mathrm{def}}{=} h_1^{(\alpha_1)} \cdots h_L^{(\alpha_L)}|0\rangle \equiv |\alpha_1\rangle \cdots |\alpha_L\rangle \tag{4}$$

where $|0\rangle$ is now the global vacuum, and the parity of the above state vector is simply given by $\sum_j p(\alpha_j)$.

The Hamiltonian operator $\mathcal{H}$ is a global operator, defined on the whole lattice. As specified in the introduction, we limit our analysis to the case in which the Hamiltonian is a sum of local operators $\mathcal{H}_j$, the latter acting on a vector space $V^{(n+1)}$ which is the tensor product of $n+1$ copies of V on $n+1$ ($n+1 < L$) neighbouring sites in the ordered state (4). Moreover, we require that $\mathcal{H}_j^{(F)}$ is a sum of even products of fermionic operators, which implies that $\mathcal{H}_j$ always has parity $p = 0$. The latter choice, which is quite reasonable from the physical point of view, allows us to limit the problem of matrix representation of graded operators to that of the matrix representation of just the local Hamiltonian $\mathcal{H}_j$. In fact, by using completeness and orthogonality properties of the basis vectors (4), we can rewrite the Hamiltonian operator $\mathcal{H}$ as

$$\mathcal{H} = \sum_j \sum_{\alpha_j,\ldots,\alpha_{j+n};\, \beta_j,\ldots,\beta_{j+n}} (H_{n+1})_{\beta_j,\ldots,\beta_{j+n}}^{\alpha_j,\ldots,\alpha_{j+n}} \mathcal{O}_{\alpha_j,\ldots,\alpha_{j+n}}^{\beta_j,\ldots,\beta_{j+n}}, \tag{5}$$

where H_{n+1} is a $d^{n+1} \times d^{n+1}$ matrix representing the local Hamiltonian operator $\mathcal{H}_j$, and

$$\mathcal{O}_{\alpha_j,\ldots,\alpha_{j+n}}^{\beta_j,\ldots,\beta_{j+n}} \doteq |\alpha_j \cdots \alpha_{j+n}\rangle\langle\beta_{j+n} \cdots \beta_j| \tag{6}$$

are local projection operators acting on sites from j to $j+n$.

As expected, from (5) we have that the matrix representation H of the $d^L \times d^L$ global Hamiltonian $\mathcal{H}$, which reads

$$H = \sum_j \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{H_{(n+1)}}_{j \to j+n} \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I}, \tag{7}$$

is fully determined by the matrix representation of the local problem, H_{n+1}. Hence, starting from a given H_{n+1}, equations (5) and (7) establish that different global isomorphic spin and fermionic Hamiltonians are simply obtained by looking to different realizations of local projection operators (6). The latter contain the ultimate significance of the JWT: by realizing them as fermionic or spin projectors one obtains the corresponding local (and global) spin and fermionic Hamiltonian. In the next section, we shall construct these operators (6) in terms of on-site projector operators, which will then be implemented explicitly in spin and fermionic realizations.

The above formulation of a quantum problem somehow reverses the standard approach, which is to describe a physical problem through an Hamiltonian operator, and successively to look for its matrix representation in order to solve it. In contrast, here we start from a matrix, which is the unique representation of a given *abstract* quantum problem—identified by the Hamiltonian (5), and by the abstract projection operators (6)—and look for its realization into different operator languages, i.e. into different physical problems.

Note that such strategy would also hold for mapping to operator languages other than spin or fermionic (for instance, anyons or hard-core bosons) which can represent the (even) Hamiltonian (5); the relevant point being the realization of the local projectors (6) in the chosen language.

Note also that the local algebra obeyed by projectors (6) is independent of the realization chosen. It reads

$$\mathcal{O}_{\alpha_j,\ldots,\alpha_{j+n}}^{\beta_j,\ldots,\beta_{j+n}}\mathcal{O}_{\alpha'_j,\ldots,\alpha'_{j+n}}^{\beta'_j,\ldots,\beta'_{j+n}} = \delta_{\alpha'_j,\beta_j}\cdots\delta_{\alpha'_{j+n},\beta_{j+n}}\mathcal{O}_{\alpha_j,\ldots,\alpha_{j+n}}^{\beta'_j,\ldots,\beta'_{j+n}}. \tag{8}$$

For n and d given, the above relations close in a (sub)algebra (of) $u(d^{n+1})$. Such local algebra is characteristic of the abstract problem, and it is realized through spin or fermionic operators when expressing the even local operators $\mathcal{O}$ in terms of spin or fermionic projectors.

3. Hamiltonian in terms of on-site projectors

Let us introduce the projection operators at site j, $\mathcal{E}_\alpha^\beta \doteq |\alpha\rangle\langle\beta|$. The operator in (6) can always be expressed as product of local projection operators $\mathcal{E}_\alpha^\beta$. This is obtained by combining together bras and kets at the same site, which, due to the possible grading of the local vectors, requires some algebra. In order to avoid cumbersome notation, we limit our analysis to the case of a Hamiltonian which describes just two-site interaction terms, the sites being at a distance 1; such an assumption in fact limits the following analysis to the one-dimensional case with nearest-neighbour interaction, though it is easily generalizable to higher dimension, and to more general local interactions.

In this case the two-site operator (6) simply reads

$$\mathcal{O}_{\alpha_j,\alpha_{j+1}}^{\beta_j,\beta_{j+1}} = (-)^{p(\beta_j)[p(\alpha_{j+1})+p(\beta_{j+1})]}\mathcal{E}_{\alpha_j}^{\beta_j}\mathcal{E}_{\alpha_{j+1}}^{\beta_{j+1}}, \tag{9}$$

where the sign in front of the on-site projectors in fact can be negative only for a graded realization of operator (6) (for instance, in the fermionic case). By explicitly implementing (9) in the spin and fermionic case, for *any dimension d* of the on-site vector space, one finally obtains the local mapping between the product of on-site projection operators at j and $j+1$ in the two cases, which we may define as *local* (generalized) JWT. It reads

$$(\mathcal{E}_S)_{\alpha_j}^{\beta_j}(\mathcal{E}_S)_{\alpha_{j+1}}^{\beta_{j+1}} \to (-)^{p(\beta_j)[p(\alpha_{j+1})+p(\beta_{j+1})]}(\mathcal{E}_F)_{\alpha_j}^{\beta_j}(\mathcal{E}_F)_{\alpha_{j+1}}^{\beta_{j+1}}. \tag{10}$$

In practice, what is often convenient to do is:

(i) start from a known local problem in some operator language (spin or fermionic);

(ii) rewrite it in terms of on-site projection operator in the same language; incidentally, this step gives the non-vanishing elements of the local matrix H_2;
(iii) use (10) to map products of on-site projectors in the other language; and finally
(iv) look for the realization of the projectors in the other language (fermionic or spin): this will be the local Hamiltonian in the other language.

Here, as an example, we study the generic case in $d = 2$.
We first construct the on-site projectors in terms of spin 1/2 and spinless fermions operators respectively. It is useful to cast them into 2×2 matrix $\mathcal{E}_X^{(2)}$ with operator entries. In the spin case it reads

$$\mathcal{E}_S^{(2)} = \begin{pmatrix} \frac{1}{2} - \sigma_z & \sigma_+ \\ \sigma_- & \frac{1}{2} + \sigma_z \end{pmatrix}, \tag{11}$$

where σ_α ($\alpha = +, -, z$) are the Pauli spin-1/2 operators, all with even parity, and the on-site basis is spanned by the eigenvectors of σ_z, $\left|-\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\right\rangle$ respectively. In the fermionic case the projector matrix, in terms of spinless fermion creation and annihilation operators $c^\dagger$, c, reads

$$\mathcal{E}_F^{(2)} = \begin{pmatrix} 1 - n & c^\dagger \\ c & n \end{pmatrix}, \tag{12}$$

with $n \doteq c^\dagger c$; now the parity of diagonal entries is even, whereas that of off-diagonal entries is odd (in general, for graded vector spaces $V^{(F)}$ the parity of projectors $\mathcal{E}_\alpha^\beta$ is $p(\alpha) + p(\beta)$).

The more general matrix representing an even Hermitian Hamiltonian with nearest-neighbour interaction in one dimension for $d = 2$ is given by the 4×4 matrix $H_2^{(2)}$,

$$H_2^{(2)} = \begin{pmatrix} h_{00}^{00} & 0 & 0 & h_{00}^{11} \\ 0 & h_{01}^{01} & h_{01}^{10} & 0 \\ 0 & h_{10}^{01} & h_{10}^{10} & 0 \\ h_{11}^{00} & 0 & 0 & h_{11}^{11} \end{pmatrix}, \tag{13}$$

where the eight non-vanishing entries (arbitrary, except the constraints $h_{01}^{10} = h_{10}^{01}$, and $h_{00}^{11} = h_{11}^{00}$ imposed by the hermiticity requirement) have been written as $h_{\beta_j\beta_{j+1}}^{\alpha_j\alpha_{j+1}}$ which makes it easy to compare with (5). Now any local spin = 1/2 Hamiltonian and its correspondent spinless fermion realization are obtained from (5), (9) by inserting a given matrix of the form (13) and the appropriate projectors (11) and (12) respectively.

For instance we may look at the fermionic realization of the isotropic Heisenberg Hamiltonian for spin-1/2 operators, which is a sum of local two-site operators of the form

$$\mathcal{H}_{H,j}^{(S)} = 2\bar{\sigma}_j \cdot \bar{\sigma}_{j+1}. \tag{14}$$

It can easily be verified that it corresponds to the choice $h_{00}^{00} = h_{01}^{10} = h_{11}^{11} = 1$ and $h_{00}^{11} = h_{01}^{01} = h_{10}^{10} = 0$ in (13), in which case $H_2^{(2)}$ is just the standard two-site permutation matrix; incidentally, this fact makes the model integrable. With the above specifications, $\mathcal{H}_{H,j}^{(S)}$ can now be written in terms of spin projectors, according to step (ii) of our scheme, as

$$\mathcal{H}_{H,j}^{(S)} = (\mathcal{E}_S)_{1_j}^{0_j}(\mathcal{E}_S)_{0_{j+1}}^{1_{j+1}} + (\mathcal{E}_S)_{0_j}^{1_j}(\mathcal{E}_S)_{1_{j+1}}^{0_{j+1}} + (\mathcal{E}_S)_{1_j}^{1_j}(\mathcal{E}_S)_{1_{j+1}}^{1_{j+1}} + (\mathcal{E}_S)_{0_j}^{0_j}(\mathcal{E}_S)_{0_{j+1}}^{0_{j+1}}. \tag{15}$$

By now implementing steps (iii) and (iv) of the same scheme to map bilinear products of spin into fermionic on-site projectors, and realizing the latter through (12) in terms of fermionic operators, we finally get the local Heisenberg Hamiltonian in its fermionic realization, which reads

$$\mathcal{H}_{H,j}^{(F)} = c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger + 2n_j n_{j+1} - n_j - n_{j+1}, \tag{16}$$

apart from constant terms. Of course, such local identification reflects into an identification of the global Hamiltonians as well, the latter being in perfect agreement with the results obtained by the non-local JWT.

4. Correspondence between spin-1 models and extended $t - J$ models

Here we illustrate the correspondence obtained by the scheme developed in the previous section in the case of an on-site vector space of dimension $d = 3$. Such a case is of particular interest since the models obtained in both realizations are thoroughly studied in the literature, exhibiting a rich structure of the quantum phase diagram; in the fermionic case, these are extended $t - J$ models with constrained fermions [6] (in particular, $t - J$ model and infinite U Hubbard model), whereas in the spin realization they correspond to spin-1 models.

In order to perform the local mapping, we have first to give the on-site projectors in the two realizations. This requires us to specify which are the basis vectors of the on-site vector space $|\alpha_R\rangle$ in the two cases, as well as how each of them correspond to a different abstract α.

For instance, in the previous example with $d = 2$ we implicitly assumed that the state $|-1/2\rangle$ in the spin realization was implemented as the empty state in the spinless fermion realization. Of course we could have chosen the opposite way, associating the empty state with the state of $|1/2\rangle$ in the spin realization; if that was the case, we would have obtained a particle–hole transform of the fermionic model corresponding to the *same* spin model, meaning that such an operation (which is a mere redefinition of basis) does not change at all the spectrum of the model. In general, for $d > 2$, one may obtain—associated with different identifications of the corresponding basis vectors in the two representations—many different mappings of the same (say) spin model, whose number is given by $(d - n_1)!n_1!$, n_1 being the number of odd-parity states (i.e. the dimension of $V^{(1)}$).

In view of the above, we first proceed to the separate construction of projection operators in the two realizations, and only after we eventually specify a correspondence between state vectors in the two realizations. In the spin case, we choose $|1_S\rangle = |1\rangle$, $|2_S\rangle = |0\rangle$, and $|3_S\rangle = |-1\rangle$, where the index on the rhs refers to the eigenvalue of S_z. With this choice the projector matrix in the spin-1 realization reads

$$\mathcal{E}_S^{(3)} = \frac{1}{2}\begin{pmatrix} S_z^2 + S_z & \sqrt{2}S_zS_+ & S_+^2 \\ \sqrt{2}S_-S_z & 2(1 - S_z^2) & -\sqrt{2}S_+S_z \\ S_-^2 & -\sqrt{2}S_zS_- & S_z^2 - S_z \end{pmatrix}. \tag{17}$$

For the fermionic case, we may identify the basis of the on-site fermionic Hilbert space with the three possible physical states $|1_F\rangle = |\uparrow\rangle$, $|2_F\rangle = |0\rangle$, $|3_F\rangle = |\downarrow\rangle$ (with $p(2_F) = 0$, $p(1_F) = p(3_F) = 1$), since two fermions on the same site are not allowed (constrained fermions); with this choice, the projection operators $\mathcal{E}_{F\alpha}^{\beta}$, which turn out to be a subset of so-called Hubbard projectors, can be cast again in the form of a 3×3 matrix $\mathcal{E}_F^{(3)}$ with operator entries; explicitly,

$$\mathcal{E}_F^{(3)} = \begin{pmatrix} \tilde{n}_\uparrow & \tilde{c}_\uparrow^\dagger & \tilde{c}_\uparrow^\dagger\tilde{c}_\downarrow \\ \tilde{c}_\uparrow & 1 - \tilde{n}_\uparrow - \tilde{n}_\downarrow & \tilde{c}\downarrow \\ \tilde{c}_\downarrow^\dagger\tilde{c}_\uparrow & \tilde{c}_\downarrow^\dagger & \tilde{n}_\downarrow \end{pmatrix}, \tag{18}$$

where, as usual, we have introduced the constraint of no double occupation through the constrained fermion operator $\tilde{c}_\sigma \doteq (1 - n_{\bar{\sigma}})c_\sigma$, with $\bar{\sigma} = -\sigma$; moreover $\tilde{n}_\sigma \doteq \tilde{c}_\sigma^\dagger\tilde{c}_\sigma$. The more general constrained fermions Hamiltonian with nearest-neighbour interaction is the

$t - J - V$ Hamiltonian [6]

$$H^{(F)}_{tJV} = -t \sum_{j,\sigma} \left(\tilde{c}^{\dagger}_{j,\sigma} \tilde{c}_{j+1,\sigma} + \text{h.c.}\right) + V \sum_{j} \tilde{n}_j \tilde{n}_{j+1} + J \sum_{j} \vec{\mathcal{S}}_j \cdot \vec{\mathcal{S}}_{j+1}, \tag{19}$$

where the standard on-site ($su(2)$) spin operator $\vec{\mathcal{S}}_j$ has been introduced: $\mathcal{S}_{+,j} = \tilde{c}^{\dagger}_{\uparrow,j} \tilde{c}_{\downarrow,j}$, $\mathcal{S}_{-,j} = \mathcal{S}^{\dagger}_{+,j}$, $\mathcal{S}_{z,j} = \frac{1}{2}(\tilde{n}_{\uparrow,j} - \tilde{n}_{\downarrow,j})$. In (19) terms not conserving the total number of electrons $\mathcal{N} \doteq \sum_j \tilde{n}_j$ and the total spin operator $\vec{\mathcal{S}} \doteq \sum_j \vec{\mathcal{S}}_j$ have been neglected.

Interestingly, $H^{(F)}_{tJV}$ reduces to the infinite U Hubbard model for $J = V = 0$, and to the standard $t - J$ model for $V = -\frac{J}{4}$, both of which have been widely investigated in the literature; in particular the exact analytical solution is known in one dimension both for the infinite U Hubbard model [7] and for the supersymmetric (i.e. $J = -2t$) $t - J$ model [6, 8].

The spin-1 realization of the $t - J - V$ Hamiltonian (19) is now obtained by specifying which α_S corresponds to a given α_F; we choose $|\alpha\rangle_F \to |\alpha\rangle_S$, in which case—up to conserved quantities—the local JWT (9) gives

$$\begin{aligned} H^{(S)}_{tJV,j} = & -t[\mathbf{S}_j\mathbf{S}_{j+1} + (\mathbf{S}_j\mathbf{S}_{j+1})^2] + \frac{2V + 3t}{2} S^2_{z,j} S^2_{z,j+1} \\ & + \frac{J + 2t}{8} \left[S^2_{+,j} S^2_{-,j+1} + S^2_{-,j} S^2_{+,j+1} + 2S_{z,j} S_{z,j+1}\right] \end{aligned} \tag{20}$$

from which the global Hamiltonian $\mathcal{H}^{(S)}_{tJV}$ is straightforwardly obtained.

The extended $t - J$ Hamiltonian in the spin-1 realization can be recognized as a sum of three independent contributions. As a general comment, one may observe that the interplay of such contributions to determine the ground-state phase diagram properties of $\mathcal{H}^{(S)}_{tJV}$ are easily deduced from those of the corresponding fermionic model [9]. In particular, the J term is expected to drive phase separation, whereas the V term is expected to be responsible of the opening of a (spin) gapped phase. These phases are now to be interpreted as driven from quadrupolar interaction in the spin description, and in particular the gapped phase should be analysed in terms of some unusual quadrupolar ordering [10].

More specific interesting observations are now in order. First of all—for arbitrary values of the three independent parameters—$\mathcal{H}^{(S)}_{tJV}$ inherits all the symmetries of its fermionic partner. Since these were built in the even sector of the on-site fermionic algebra, in order to give them we simply have to translate $\mathcal{N}$ and $\vec{\mathcal{S}}$ in terms of on-site projectors, and then to rewrite the latter into their spin realization. Explicitly, they turn out to be

$$\mathcal{N} = \sum_j S^2_{z,j}, \qquad \mathcal{S}_+ = \sum_j S^2_{+,j}, \qquad \mathcal{S}_- = \mathcal{S}^{\dagger}_+, \qquad \text{and} \qquad \mathcal{S}_z = \frac{1}{2} \sum_j S_{z,j}.$$

Also, we note that the choice $J = -2t$, $V = -\frac{3}{2}t$, which in the spin realization corresponds to the pure bilinear biquadratic spin-1 Hamiltonian with $\Delta = 1$ [10], in the fermionic realization reads as a $t - J$ supersymmetric model extended by a nearest-neighbour repulsive term, in perfect agreement with [1]; in this case the absolute ground state of the latter coincides with that given by Sutherland in [8] for the $SU(3)$ symmetric spin-1 model (also known as the F^3 case (see below)), and the spectrum is gapless. Even more interestingly, there are other choices of parameters in (19) corresponding to integrable cases which generate integrable spin-1 models not discussed in the literature.

First of all, the supersymmetric $t - J$ model in the spin realization reads

$$H^{(S)}_{tJss} = -t \sum_j \left[\mathbf{S}_j\mathbf{S}_{j+1} + (\mathbf{S}_j\mathbf{S}_{j+1})^2 - 2S^2_{z,j} S^2_{z,j+1}\right], \tag{21}$$

this is the bilinear biquadratic $\Delta = 1$ spin Hamiltonian already cited, extended by a diagonal nearest-neighbour quadrupolar interaction. The ground state of the latter is hence given by

Lai–Sutherland [8, 11] solution of supersymmetric $t-J$ model, and the spectrum is thoroughly discussed in the literature.

Also surprising is the choice $J = V = 0$, which in the fermionic realization would correspond to the infinite U Hubbard model: its spectrum in one dimension is known [7] to be that of a spinless fermion; in the spin-1 realization we have that such is also the spectrum of a non-trivial model which to our knowledge was never proved to be integrable.

5. Integrable spin-1 models as generalized permutators

Let us exploit more closely the above observations. Thanks to the local character of our JWT, models proved to be integrable in one language—which feature refers to the structure of the local Hamiltonian—are straightforwardly translated into integrable models in the other language. In the following, we provide other integrable spin-1 models starting from fermionic ones.

It has recently been shown [5] that both supersymmetric $t - J$ and infinite U electron models are integrable in one dimension since they belong to a larger class of models for which the local Hamiltonians have the structure of *generalized permutators*.

The meaning of a generalized permutator is easily understood in terms of so-called Sutherland species (SS). Starting from the on-site vector space V, we may think to group its d basis vector (or *physical species*) into $N_S \leqslant d$ different species which are called the Sutherland species. Each of these species is left unchanged (apart from a possible sign change, see below) by the action of the generalized permutator, the latter interchanging only basis vector belonging to different SSs. A generalized permutator would then have the structure of an ordinary permutator if represented on a local vector space of dimension N_S.

Each of the N_S SS is said to be bosonic (B) if no sign change occurs after action of permutator, or fermionic (F) otherwise. For arbitrary d, there are many possibilities of grouping the d physical species into N_S SS; each of them corresponding to a different generalized permutator. The latter can be classified as $B^l F^{k-l}$ for $0 \leqslant l \leqslant k$, $1 < k \leqslant N_S$. Whenever the matrix representing the two-site Hamiltonian coincides with the matrix representation of a generalized permutator, the global Hamiltonian is integrable.

Returning to our case, for which $d = 3$, it can be seen [5] that the infinite U Hubbard model is a BF model, where the bosonic species at each site is the vacuum, and the fermionic one is formed by the two singly occupied states (with up and down spin), whereas the supersymmetric $t-J$ model is a BF^2 model, the three SS coinciding precisely with the physical species at each site $|\alpha_F\rangle$. In general, it has been shown that all integrable fermionic models with an on-site vector space V of dimension 3, which locally act as generalized permutator [12], and globally preserve $\vec{\mathcal{S}}$ and $\mathcal{N}$ are eight; in correspondence to the possible different choices $J = (s_1+s_2)t$, and $V = -\frac{J}{4} + (s_1 + s_3)t$; with $s_\alpha = \pm 1$ for $\alpha = 1, 2, 3$ independent signs. Our spin–fermion mapping now allows us to map the integrable fermionic cases into spin 1 ones; interestingly, apart from the cases with $J = \pm 2t$ already discussed in the previous section, in so doing we obtain other four integrable spin-1 models, which fact to our knowledge was never noted. All of these imply the choice $J = 0$, thus corresponding to generalizations of the infinite U Hubbard model. In the spin realization their Hamiltonian reads

$$H^{(S)}_{EU\infty} = -t \sum_j \left\{ [(S_+S_z)_j(S_zS_-)_{j+1} + (S_zS_+)_j(S_-S_z)_{j+1} + \text{h.c.}] + (s_1+s_2)S^2_{z,j}S^2_{z,j+1} \right\}. \quad (22)$$

We can provide ground-state energy $\epsilon = E_0/L$, with E_0 lowest eigenvalue, for each of these models.

In fact, following Sutherland notation, it is easily seen that the models coincide—up to conserved quantities—with a B^2 model for $s_1 = s_2 = -1$, with two BF models for $s_1 = -s_2$, and with a F^2 model for $s_1 = s_2 = +1$. For all of them, ϵ can be evaluated using the extension of Sutherland theorem which holds for generalized permutators [13], and turns out to be

$$\epsilon = \begin{cases} -1 & \text{for} \quad B^2 \\ 2n_F - 1 - \dfrac{2}{\pi}\sin \pi n_F & \text{for} \quad \mathrm{BF} \\ 1 - 2\ln 2 & \text{for} \quad F^2; \end{cases} \tag{23}$$

here $n_F = N_F/L$ is the density of the fermionic species, which is related to $\mathcal{N}$: $N_F = \mathcal{N}$ for $s_1 = +1$, $N_F = L - \mathcal{N}$ for $s_1 = -1$.

6. Summary and conclusions

In this paper we revisited known spin–fermion mappings showing that the underlying structure is that of local identities, which relate different realizations of abstract projection operators. This result is contained in equations (5) and (6) and holds for arbitrary dimension of both the ambient lattice Λ (D) and the on-site vector space V (d), under the very reasonable condition that the physical Hamiltonian is an even operator in the fermionic fields. We then specialized to one-dimensional lattice, and in this case we explicitly gave our generalized JWT in terms of simple local relations between bilinear products of on-site projection operators in the spin and fermionic languages (equation 10). The latter still holds for arbitrary d, giving the standard JWT results for $d = 2$. Finally we focused on the case $d = 3$, obtaining from the extended fermionic $t - J$ models, the corresponding $S = 1$ models. In particular, we used the mapping to generate new integrable spin-1 cases (equations (21) and (22)) providing for each of them the ground-state energy.

Possible developments of the present work are on the one hand in the generalization of the local identity to other particle realizations (for instance, hard-core bosons and anyons), as well as in the explicit analysis of cases with $d > 3$ (the correspondence of extended Hubbard models with spin-3/2 models being of course the easier application), and on the other hand in providing a bridge to comprehension/solutions of models which could strongly differ in their physical meaning, but are unified from a mathematical point of view: for instance, even models in dimension greater than one, such as strips and ladders, can be mapped into one-dimensional models with on-site vector space of appropriate (finite) dimension avoiding the sign problem. Work is in progress along these lines.

References

[1] Batista C D and Ortiz G 2001 *Phys. Rev. Lett.* **86** 1082
[2] Batista C D and Ortiz G 2004 *Adv. Phys.* **53** 1
[3] Jordan P and Wigner E 1928 *Z. Phys.* **47** 631
[4] Korepin V E, Bogolubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[5] Dolcini F and Montorsi A 2001 *Nucl. Phys.* B **592** 563
[6] Schlottmann P 1987 *Phys. Rev.* B **36** 5177
[7] Caspers W J and Iske P I 1989 *Physica* A **157** 1033
Kotrla M 1990 *Phys. Lett.* A **145** 33
[8] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[9] Troyer M *et al* 1993 *Phys. Rev.* B **48** 4002
[10] Solyom J 1987 *Phys. Rev.* B **36** 8642
[11] Lai C K 1974 *J. Math. Phys.* **15** 1675
[12] Anfossi A, Dolcini F and Montorsi A 2004 *Recent Research Developements in Physics* vol 5 (Trivandrum: Transworld Research Network) p 513
[13] Dolcini F and Montorsi A 2001 *Phys. Rev.* B **63** (R)121103